NON-COMMUTATIVE CASTELNUOVO-MUMFORD REGULARITY AND AS-REGULAR ALGEBRAS

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Abstract. Let A be a connected graded k-algebra with a balanced dualizing complex. We prove that A is a Koszul AS-regular algebra if and only if that the Castelnuovo-Mumford regularity and the Ext-regularity coincide for all finitely generated A-modules. This can be viewed as a non-commutative version of [Rö, Theorem 1.3]. By using Castelnuovo-Mumford regularity, we prove that any Koszul standard AS-Gorenstein algebra is AS-regular. As a preparation to prove the main result, we also prove the following statements are equivalent: (1) A is AS-Gorenstein; (2) A has finite left injective dimension; (3) the dualizing complex has finite left projective dimension. This generalizes [Mo, Corollary 5.9].

1. Introduction

A coherent sheaf $\mathcal{M}$ on $\mathbb{P}^n$ is called m-regular if $H^i(\mathcal{M}(m-i)) = 0$ for all $i \geq 1$. Mumford proved a vanishing theorem [Mu]: if $\mathcal{M}$ is m-regular, then it is $\overline{m}$-regular for all $\overline{m} > m$. Motivated by Mumford's vanishing theorem, a notion of regularity was introduced by Eisenbud and Goto [EG] for graded modules over polynomial algebras. This notion of regularity is closely related to the regularity of sheaves, also to the existence of linear free resolutions of the truncations of graded modules and to the degrees of generators of the syzygies. If $A = k[x_1, x_2, \cdots, x_{n+1}]$, so that Proj $A = \mathbb{P}^n$, and M is an A-module of the form $\oplus_{i\geq 0} H^0(\mathbb{P}^n, \mathcal{O}_X(i))$ for some scheme $X \subset \mathbb{P}^n$, then the regularity of M is the regularity of $X \subset \mathbb{P}^n$ in the sense of Castelnuovo, studied by Mumford [Mu] and many other authors in literature. There are several competing definitions of Castelnuovo-Mumford regularity for a graded module over a commutative or non-commutative connected graded k-algebra A, say, $\mathrm{CM.reg}_A M$ defined by using local cohomology (Definition 4.1 and Remark 4.2), $\mathrm{Ext.reg}_A M$ defined by Ext-group (Definition 4.4) and $\mathrm{Tor.reg}_A M$ defined by Tor-group (Remark 4.5). The Ext-regularity and Tor-regularity of a finitely generated graded module are always the same. If A is a polynomial algebra with standard grading, Eisenbud and Goto [EG] proved that CM.reg M = Tor.reg M for all non-zero finitely generated graded A-modules. Römer [Rö] proved that the converse is true, i.e., if A is a commutative connected graded k-algebra generated in degree 1, such that CM.reg M = Tor.reg M holds for all non-zero finitely generated

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graded A-modules, then A is a polynomial algebra with standard grading (see Theorem 5.1). The non-commutative Castelnuovo-Mumford regularity was first studied by Jørgensen [Jo3, Jo4]. Jørgensen proved a version of Mumford's vanishing theorem for non-commutative projective schemes [Jo3, Theorem 2.4 and Corollary 3.3]. Under the assumption that A is a Noetherian connected graded k-algebra having a balanced dualizing complex, Jørgensen proved that for any finitely generated non-zero A-module M, $-\text{CM.reg } A \leq \text{Ext.reg } M - \text{CM.reg } M \leq \text{Ext.reg } k$ [Jo4, Theorem 2.5 ,2.6]. In the commutative case, this was also proved by Römer [Rö, Theorem 1.2]. For a Noetherian connected graded k-algebra A, by a result of Van den Bergh [VdB, Theorem 6.3], A has a balanced dualizing complex if and only if that A has finite left and right local cohomology dimension (see Definition 2.3) and A satisfies the left and right χ-condition (see Definition 2.4). The χ-condition is trivially true when A is commutative. If A is commutative, then the Krull dimension of A is finite. By a theorem of Grothendieck (see [BH, Theorem 3.5.7]), the local cohomology dimension of A is finite. These conditions are very natural and important in non-commutative projective geometry [AZ]. Under these assumptions, in this article, we will prove a non-commutative version (Theorem 5.4) of Römer's result [Rö, Theorem 1.3].

Theorem. Let A be a Noetherian connected graded k-algebra having a balanced dualizing complex. Then A is Koszul AS-regular if and only if $\text{CM.reg } M = \text{Ext.reg } M$ holds for all non-zero finitely generated graded A-modules.

Our proof is different from Römer's in the commutative case. Römer used the following fact in his proof: every commutative Noetherian connected graded k-algebra can be viewed as a graded quotient of some polynomial algebra. However, in the non-commutative case, it is still an unsolved problem whether a Noetherian connected graded k-algebra with a balanced dualizing complex can be viewed as a graded quotient of an AS-Gorenstein algebra. Actually we will prove that (iii)$\Leftrightarrow$(iv) $\Leftrightarrow$(v) in [Rö, Theorem 4.1] (see also Theorem 5.1) holds in the non-commutative case. Note that (i)$\Leftrightarrow$(ii) is always true. However, (i)$\Leftrightarrow$(iii) does not hold in general (see Remark 5.5). Since in the commutative case, the following crucial fact is used in the Römer's proof: Let A be a commutative Noetherian connected graded k-algebra generated in degree 1, then A is Koszul if and only if $\text{Ext.reg } k < \infty$. This was first a conjecture in [AE], and finally proved by L. L. Avramov and I. Peeva in [AP]. However, this fact is not true in the non-commutative case since $\text{Ext.reg } k < \infty$ holds for those non-Koszul AS-regular algebras (see Proposition 5.6 and Remark 5.5).

Before giving the proof of the main result, we prove the following result (Theorem 4.11) by using Catelnuovo-Mumford regularity, which is of independent interest.

Theorem. Any Koszul standard AS-Gorenstein algebra is AS-regular.

An AS-Gorenstein algebra of type (d, l) is called standard if $d = l$ (Definition 3.1).

We also generalize a result of Mori [Mo, Corollary 5.9] as a preparation to the proof of the main result (Theorem 3.5).

Theorem. Let A be a connected graded k-algebra with a balanced dualizing complex. Then A is AS-Gorenstein if and only if that A has finite left injective resolution, if and only if that the dualizing complex has finite left projective dimension.

2. Notations and Preliminaries

Throughout this article, k is a fixed field. A k-algebra A is called an $\mathbb{N}$-*graded k-algebra* if A, as a k-vector space, has the form $A = \bigoplus_{i\geq 0} A_i$ such that $A_iA_j \subseteq A_{i+j}$ for all $i, j \geq 0$. If further, $A_0 = k$, then A is called *connected graded*. A left A-module M is called a *graded A-module* if M, as a k-vector space, has the form $M = \bigoplus_{i=-\infty}^{\infty} M_i$ such that $A_iM_j \subseteq M_{i+j}$ for all $i \geq 0$ and $j \in \mathbb{Z}$. Right graded A-modules are defined similarly.

The opposite algebra of A is denoted by A^{o}; it is the same as A as k-vector spaces, but the product is given by $a \cdot b = ba$. A right graded A-module can be identified with a left graded A^{o}-module. Unless otherwise stated, we are working with left modules. We use the term Noetherian for two-sided Noetherian.

Let $A = \bigoplus_{i\geq 0} A_i$ be an $\mathbb{N}$-graded k-algebra. We write $\mathfrak{m} = \bigoplus_{i\geq 1} A_i$. When A is connected graded, the graded module ${}_Ak \cong A/\mathfrak{m}$ is called the *trivial module*. A graded module $M = \bigoplus_{i=-\infty}^{\infty} M_i$ is called *left-bounded* if $M_i = 0$ for $i \ll 0$; *right-boundedness* and *boundedness* are defined similarly. M is called *locally finite* if each graded piece M_i is a finite dimensional k-vector space. For any fixed $n \in \mathbb{Z}$, we use the notations $M_{>n} = \bigoplus_{i=n+1}^{\infty} M_i$, $M_{<n} = \bigoplus_{i=-\infty}^{n-1} M_i$ and $M_{\neq n} = \bigoplus_{i\neq n} M_i$. For any graded A-module M, the n-th *shift* $M(n)$ of M, is defined by $M(n)_i = M_{n+i}$.

Let M and N be graded A-modules. A left A-module homomorphism $f : M \to N$ is said to be *a graded homomorphism of degree l* if $f(M_i) \subseteq N_{i+l}$ for all $i \in \mathbb{Z}$. Let $\mathrm{Gr}\, A$ denote the category of graded left A-modules and graded homomorphisms of degree 0. Let $\mathrm{gr}\, A$ be the full subcategory of $\mathrm{Gr}\, A$ consisting of finitely generated graded A-modules. Let $\mathrm{Hom}_{\mathrm{Gr}\, A}(-,-)$ be the homo-functor in the category $\mathrm{Gr}\, A$.

Let A and B be $\mathbb{N}$-graded k-algebras. The category $\mathrm{Gr}\, A^{\mathrm{o}}$ is identified with the category of graded right A-modules; the category $\mathrm{Gr}\, A\otimes_k B^{\mathrm{o}}$ is identified with the category of graded A-B-bimodules. In particular, the category of graded A-A-bimodules is denoted by $\mathrm{Gr}\, A^e$, where $A^e = A\otimes_k A^{\mathrm{o}}$. The natural restriction functors are denoted by

$$\mathrm{res}_A : \mathrm{Gr}\ A \otimes_k B^{\mathrm{o}} \longrightarrow \mathrm{Gr}\ A$$

and

$$\mathrm{res}_{B^{\mathrm{o}}} : \mathrm{Gr}\ A \otimes_k B^{\mathrm{o}} \longrightarrow \mathrm{Gr}\ B^{\mathrm{o}}.$$

We define the graded Hom-functor by

$$\underline{\mathrm{Hom}}_A(M, N) = \bigoplus_{n=-\infty}^{\infty} \mathrm{Hom}_{\mathrm{Gr\, A}}(M, N(n)).$$

$\underline{\mathrm{Hom}}_A(M,N)$ is naturally a graded left B-module if M is a graded A-B-bimodule and a graded right C-module if N is a graded A-C-bimodule.

For any $M \in \mathrm{Gr}\, A \otimes_k B^{\mathrm{o}}$, $M' = \underline{\mathrm{Hom}}_k(M,k)$ is called its *Matlis dual*. By definition, M' becomes a graded B-A-bimodule. Thus there is an exact contravariant functor from $\mathrm{Gr}\, A \otimes_k B^{\mathrm{o}}$ into $\mathrm{Gr}\, B \otimes_k A^{\mathrm{o}}$. If M is locally finite, then $M'' \cong M$ as A-B-bimodules.

Let $X^{\cdot}, Y^{\cdot}$ be cochain complexes of graded left A-modules. The i-th cohomology module of $X^{\cdot}$ is denoted by $h^i(X^{\cdot})$. A morphism of complexes $f : X^{\cdot} \longrightarrow Y^{\cdot}$ is called a *quasi-isomorphism* if $h^i(f) : h^i(X^{\cdot}) \longrightarrow h^i(Y^{\cdot})$ is an isomorphism for all $i \in \mathbb{Z}$. Shifting of complexes is denoted by $[\ ]$ so that $(X^{\cdot}[n])^p = X^{n+p}$. We denote by $Z^n(X^{\cdot}) = \mathrm{Ker}(d^n_{X^{\cdot}})$ and $B^n(X^{\cdot}) = \mathrm{Im}(d^{n-1}_{X^{\cdot}})$. Let $X^{\geq n}$ denote the (brutal) truncated complex

$$0 \longrightarrow X^n \longrightarrow X^{n+1} \longrightarrow \cdots,$$

and let $X^{\leq n}$ denote the complex

$$\cdots \longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow 0.$$

The homotopy category of $\mathrm{Gr}\, A$ is denoted by $K(\mathrm{Gr}\, A)$ and the derived category of $\mathrm{Gr}\, A$ is denoted by $\mathrm{D}(\mathrm{Gr}\, A)$. For any $X^{\cdot} \in \mathrm{D}(\mathrm{Gr}\, A)$, we define

$$\sup X^{\cdot} = \sup\{i \mid h^i(X^{\cdot}) \neq 0\} \text{ and } \inf X^{\cdot} = \inf\{i \mid h^i(X^{\cdot}) \neq 0\}.$$

When $\sup X^{\cdot} < \infty$(resp. $\inf X^{\cdot} > -\infty$), $X^{\cdot}$ is said to be *bounded above* (resp. *bounded below*). $X^{\cdot}$ is said to be *bounded* if $X^{\cdot}$ is both bounded above and bounded below. In this case, $\mathrm{amp}\, X^{\cdot} = \sup X^{\cdot} - \inf X^{\cdot}$ is called the *amplitude* of $X^{\cdot}$. There are various full subcategories $\mathrm{D}^*(\mathrm{Gr}\, A)$ of $\mathrm{D}(\mathrm{Gr}\, A)$, where $* = -, +, b$, consisting of bounded above, bounded below and bounded complexes respectively. There are also full subcategories $\mathrm{D}_*(\mathrm{Gr}\, A)$ of $\mathrm{D}(\mathrm{Gr}\, A)$, where $* = \mathrm{fg}, \mathrm{lf}$, consisting of complexes with finitely generated cohomologies and locally finite cohomologies respectively. Super- and subscripts are combined freely.

Let A, B, and C be $\mathbb{N}$-graded k-algebras. For any $X^{\cdot} \in K(A \otimes_k B^{\mathrm{o}})$ and any $Y^{\cdot} \in K(A \otimes_k C^{\mathrm{o}})$, $\mathrm{Hom}^{\cdot}_A(X^{\cdot}, Y^{\cdot})$ is a complex in $K(B \otimes_k C^{\mathrm{o}})$ where the n-th term is

$$\mathrm{Hom}^n_A(X^{\cdot}, Y^{\cdot}) = \prod_p \underline{\mathrm{Hom}}^n_A(X^p, Y^{p+n}),$$

and the differential $d^n_{\mathrm{Hom}^{\cdot}_A(X^{\cdot},Y^{\cdot})}$ is

$$(f_p)_p \rightsquigarrow (f_{p+1} \cdot d^p_{X^{\cdot}} - (-1)^n d^{p+n}_{Y^{\cdot}} \cdot f_p)_p.$$

This induces a bi-∂-functor,

$$\mathrm{Hom}^{\cdot}_A(-,-) : \quad K(A \otimes_k B^{\mathrm{o}})^{\mathrm{o}} \times K(A \otimes_k C^{\mathrm{o}}) \longrightarrow K(B \otimes_k C^{\mathrm{o}}).$$

The right-derived functor of $\mathrm{Hom}^{\cdot}$ is denoted by $R\underline{\mathrm{Hom}}$.

Similarly, for any $X^{\cdot} \in K(B \otimes_k A^{\mathrm{o}})$ and $Y^{\cdot} \in K(A \otimes_k C^{\mathrm{o}})$, $X^{\cdot} \otimes_A Y^{\cdot} \in K(B \otimes_k C^{\mathrm{o}})$, where the n-th component $X^{\cdot} \otimes_A^n Y^{\cdot}$ of $X^{\cdot} \otimes_A Y^{\cdot}$ is the k-vector subspace generated by

$$\{x^p \otimes y^q \mid x^p \in X^p, y^q \in Y^q, p+q=n\}$$

and the tensor differential $d_{X^{\cdot} \otimes_A Y^{\cdot}} = d_{X^{\cdot}} \otimes \mathrm{id}_Y + \mathrm{id}_X \otimes d_{Y^{\cdot}}$ (with Koszul sign rule). This induces a bi-∂-functor,

$$- \otimes_A - : \quad K(B \otimes_k A^{\mathrm{o}}) \times K(A \otimes_k C^{\mathrm{o}}) \longrightarrow K(B \otimes_k C^{\mathrm{o}}).$$

The left derived functor of $\otimes$ is denoted by ${}^{L}\otimes$.

The Ext and Tor are defined as

$$\underline{\mathrm{Ext}}_A^i(X^{\cdot}, Y^{\cdot}) = h^i(R\underline{\mathrm{Hom}}_A(X^{\cdot}, Y^{\cdot})) \quad \text{and} \quad \mathrm{Tor}_i^A(X^{\cdot}, Y^{\cdot}) = h^{-i}(X^{\cdot} {}^{L}\otimes_A Y^{\cdot}).$$

Definition 2.1. Let A be an $\mathbb{N}$-graded k-algebra and let $X^{\cdot} \in \mathrm{D}^-(\mathrm{Gr}\, A)$. A projective resolution of $X^{\cdot}$ is a complex $P^{\cdot}$ consisting of projective modules such that there is a quasi-isomorphism $f : P^{\cdot} \xrightarrow{\simeq} X^{\cdot}$. Moreover, if $B^i(P^{\cdot}) \subseteq \mathfrak{m}P^i$ for each $i \in \mathbb{Z}$, then $P^{\cdot}$ is called a minimal projective resolution of $X^{\cdot}$.

We call

$$\mathrm{pd}_A(X^{\cdot}) = \inf_{P^{\cdot}}(-\inf\{i \mid P^i \neq 0\})$$

the projective dimension of $X^{\cdot}$, where the infimum is taken over all projective resolutions $P^{\cdot}$ of $X^{\cdot}$.

If A is left Noetherian and $X^{\cdot} \in \mathrm{D}^-_{fg}(\mathrm{Gr}\, A)$, then

$$\mathrm{pd}_A(X^{\cdot}) = \sup\{i \mid \underline{\mathrm{Ext}}_A^i(X^{\cdot}, M) \neq 0 \text{ for some } M \in \mathrm{gr}\, A\}.$$

If $P^{\cdot} \xrightarrow{\simeq} X^{\cdot}$ is a minimal free resolution of $X^{\cdot}$, then $\mathrm{pd}_A(X^{\cdot}) = -\inf\{i \mid P^i \neq 0\}$.

Definition 2.2. Let A be an $\mathbb{N}$-graded k-algebra and let $X^{\cdot} \in \mathrm{D}^+(\mathrm{Gr}\, A)$. An injective resolution of $X^{\cdot}$ is a complex $I^{\cdot}$ consisting of injective modules such that there is a quasi-isomorphism $f : X^{\cdot} \xrightarrow{\simeq} I^{\cdot}$. Moreover, if $Z^i(I^{\cdot})$ is graded essential in I^i for each $i \in \mathbb{Z}$, then $I^{\cdot}$ is called a minimal injective resolution of $X^{\cdot}$.

We call

$$\mathrm{id}_A(X^{\cdot}) = \inf_{I^{\cdot}}(\sup\{i \mid I^i \neq 0\})$$

the injective dimension of $X^{\cdot}$, where the infimum is taken over all injective resolutions $I^{\cdot}$ of $X^{\cdot}$.

It is easy to know that

$$\mathrm{id}_A(X^{\cdot}) = \sup\{i \mid \underline{\mathrm{Ext}}_A^i(M, X^{\cdot}) \neq 0 \text{ for some } M \in \mathrm{gr}\, A\}.$$

Next we collect some definitions and facts in connected graded ring theory we need, which are of basic importance not only in this article.

Definition 2.3. [AZ, Ye] Let A be an $\mathbb{N}$-graded k-algebra and $\mathfrak{m} = A_{\geq 1}$.

(1) For any $M \in \mathrm{Gr}\, A$, the $\mathfrak{m}$-torsion submodule of M is defined to be

$$\Gamma_{\mathfrak{m}}(M) = \{x \in M \mid A_{\geq n} \cdot x = 0, \text{ for } n \gg 0\}.$$

If $\Gamma_{\mathfrak{m}}(M) = M$, then M is said to be $\mathfrak{m}$-torsion.

(2) $\Gamma_{\mathfrak{m}} : \mathrm{Gr}\, A \longrightarrow \mathrm{Gr}\, A,\ M \mapsto \Gamma_{\mathfrak{m}}(M)$, is a left exact functor. Its right derived functor $R\Gamma_{\mathfrak{m}}$ is defined on the derived category $\mathrm{D}^+(\mathrm{Gr}\ A)$. The i-th local cohomology of $X^\cdot \in \mathrm{D}^+(\mathrm{Gr}\ A)$ is defined to be

$$\mathrm{H}^i_{\mathfrak{m}}(X^\cdot) = h^i(R\Gamma_{\mathfrak{m}}(X^\cdot)).$$

(3) The local cohomological dimension of a graded A-module M is defined to be

$$\mathrm{lcd}(M) = \sup\{i \mid \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0\}.$$

(4) The cohomological dimension of $\Gamma_{\mathfrak{m}}$ is defined to be

$$\mathrm{cd}(\Gamma_{\mathfrak{m}}) = \sup\{\mathrm{lcd}(M) \mid M \in \mathrm{Gr}\, A\}.$$

(5) $\mathrm{cd}(\Gamma_{\mathfrak{m}})$ is also called the left local cohomological dimension of the algebra A.

Definition 2.4. [AZ, 3.2] Let A be a Noetherian connected graded k-algebra. Then A is said to satisfy the χ-condition if $\underline{\mathrm{Ext}}^i_A(k, M)$ is right bounded for any $M \in \mathrm{gr}\, A$ and $i \in \mathbb{Z}$.

Definition 2.5. Let A be a connected graded k-algebra. For any $X^\cdot \in \mathrm{D}^+(\mathrm{Gr}\, A)$,

$$\mathrm{depth}_A(X^\cdot) = \inf R\underline{\mathrm{Hom}}_A(k, X^\cdot) = \inf\{i \mid \underline{\mathrm{Ext}}^i_A(k, X^\cdot) \neq 0\}.$$

$\mathrm{depth}_A(X^\cdot)$ is either an integer or ∞.

Lemma 2.6. Let A be a left Noetherian connected graded k-algebra. Then for any $M \in \mathrm{Gr}\, A$, $\mathrm{depth}_A(M) = \inf\{i \in \mathbb{Z} \mid \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0\}$.

Theorem 2.7. (The Auslander-Buchsbaum formula) Let A be a left Noetherian connected graded k-algebra satisfying the χ-condition. Given any $X^\cdot \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Gr}\, A)$ with $\mathrm{pd}_A(X^\cdot) < \infty$, one has

$$\mathrm{pd}_A(X^\cdot) + \mathrm{depth}_A(X^\cdot) = \mathrm{depth}_A(A).$$

Proof. See [Jo2, Theorem 3.2]. □

Definition 2.8. [Ye, 3.3] Let A be a Noetherian connected graded k-algebra. A complex $R^\cdot \in \mathrm{D}^{\mathrm{b}}(\mathrm{Gr}\, A^e)$ is called a dualizing complex if it satisfies the following conditions:

(1) $\mathrm{id}_A(R^\cdot) < \infty$ and $\mathrm{id}_{A^\circ}(R^\cdot) < \infty$;

(2) $\mathrm{res}_A(R^\cdot) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Gr}\, A)$ and $\mathrm{res}_{A^\circ}(R^\cdot) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Gr}\, A^\circ)$;

(3) The natural morphisms $A \to R\underline{\mathrm{Hom}}_A(R^\cdot, R^\cdot)$ and $A \to R\underline{\mathrm{Hom}}_{A^\circ}(R^\cdot, R^\cdot)$ are isomorphisms in $\mathrm{D}^{\mathrm{b}}(\mathrm{Gr}\, A^e)$.

Definition 2.9. [Ye, 4.1] Let A be a Noetherian connected graded k-algebra and $R^{\cdot} \in \mathrm{D}^{\mathrm{b}}(\mathrm{Gr}\, A^e)$ be a dualizing complex over A. If there are isomorphisms $R\Gamma_{\mathfrak{m}}(R^{\cdot}) \cong R\Gamma_{\mathfrak{m}^{\circ}}(R^{\cdot}) \cong A'$ in $\mathrm{D}(\mathrm{Gr}\, A^e)$, then $R^{\cdot}$ is called balanced.

Theorem 2.10. [VdB, 6.3] Let A be a Noetherian connected graded k-algebra. Then A has a balanced dualizing complex if and only if the following two conditions are satisfied:

(1) $\mathrm{cd}(\Gamma_{\mathfrak{m}}) < \infty$, and $\mathrm{cd}(\Gamma_{\mathfrak{m}^{\circ}}) < \infty$;

(2) Both A and A° satisfy the χ-condition.

If these conditions are satisfied, then the balanced dualizing complex over A is given by $R\Gamma_{\mathfrak{m}}(A)'$.

Theorem 2.11. (The local duality theorem) Let A, C be connected graded k-algebras. Assume that A is left Noetherian with $\mathrm{cd}(\Gamma_{\mathfrak{m}}) < \infty$. Then for any $X^{\cdot} \in \mathrm{D}(\mathrm{Gr}\, A \otimes_k C^{\circ})$, there is an isomorphism

$$R\Gamma_{\mathfrak{m}}(X^{\cdot})' \cong R\underline{\mathrm{Hom}}_A(X^{\cdot}, R\Gamma_{\mathfrak{m}}(A)')$$

in $\mathrm{D}(\mathrm{Gr}\, C \otimes_k A^{\circ})$.

Proof. See [VdB, Theorem 5.1]. □

Our basic reference for homological algebra is [We].

3. AS-Gorenstein algebras

As a preparation to prove the main result (Theorem 5.4), we generalize a result of Mori [Mo, Corollary 5.9] in this section. Let's recall some definitions.

Definition 3.1. Let A be a Noetherian connected graded k-algebra. A is called left AS-Gorenstein (AS stands for Artin-Schelter) if

(1) $\mathrm{id}_A A = d < \infty$;

(2) $\underline{\mathrm{Ext}}^i_A(k, A) = \begin{cases} 0 & i \neq d \\ k(l) & i = d \end{cases}$ for some $l \in \mathbb{Z}$.

Right AS-Gorenstein algebras are defined similarly. A is AS-Gorenstein means that A is both left and right AS-Gorenstein.

Definition 3.2. Let A be a Noetherian connected graded k-algebra.

(1) A is called AS-Cohen-Macaulay if $R\Gamma_{\mathfrak{m}}(A)$ is concentrated in one degree.

(2) An A-A-bimodule ω_A is called a balanced dualizing module if $\omega_A[d]$ is a balanced dualizing complex over A for some integer d.

(3) A is called balanced Cohen-Macaulay if it has a balanced dualizing module.

Lemma 3.3. Let A be a Noetherian connected graded k-algebra with a balanced dualizing complex $R^{\cdot}$. If $\mathrm{pd}_A R^{\cdot} < \infty$, then A is AS-Cohen-Macaulay and balanced Cohen-Macaulay.

Proof. Let $F^{\cdot} \xrightarrow{\simeq} R^{\cdot}$ be a finitely generated minimal free resolution of $R^{\cdot} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Gr}\, A)$. Then

$$\mathrm{pd}_A R^{\cdot} = -\inf\{p \in \mathbb{Z} \mid F^p \neq 0\} \geq -\inf R^{\cdot}.$$

On the other hand, by the local duality theorem, $R\Gamma_{\mathfrak{m}}(k)' \cong R\underline{\mathrm{Hom}}_A(k, R^{\cdot})$ (see Theorem 2.11). Hence $\mathrm{depth}_A R^{\cdot} = 0$. It follows from Auslander-Buchsbaum formula (Theorem 2.7) that $-\inf R^{\cdot} \leq \mathrm{pd}_A R^{\cdot} = \mathrm{depth}_A A$. By Lemma 2.6 and $R^{\cdot} \cong R\Gamma_{\mathfrak{m}}(A)'$,

$$\mathrm{depth}_A A = \inf\{j \in \mathbb{Z} \mid \mathrm{H}^j_{\mathfrak{m}}(A) \neq 0\} = -\sup R^{\cdot}.$$

Therefore $\sup R^{\cdot} = \inf R^{\cdot}$, and so A is AS-Cohen-Macaulay and balanced Cohen-Macaulay. □

Lemma 3.4. Let A be a Noetherian connected graded k-algebra with a dualizing complex $R^{\cdot}$. Then for any $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Gr}\, A)$, the following holds.

(1) $\mathrm{id}_{A^\circ}(\mathrm{RHom}_A(X, R)) \leq \mathrm{pd}_A\, X + \mathrm{id}_{A^\circ}\, R$;

(2) $\mathrm{pd}_{A^\circ}(\mathrm{RHom}_A(X, R)) \leq \mathrm{id}_A\, X - \inf R$.

As a consequence, $\mathrm{pd}_A\, X < \infty$ if and only if $\mathrm{id}_{A^\circ}(\mathrm{RHom}_A(X, R)) < \infty$.

Proof. See [WZ, Lemma 2.1]. □

The following theorem says that if the balanced dualizing complex has finite projective dimension, then A is AS-Gorenstein. This is a generation of [Mo, Corollary 5.9],where A is a Noetherian balanced Cohen-Macaulay algebra. For the completeness, we give a proof here.

Theorem 3.5. Let A be a Noetherian connected graded k-algebra with a balanced dualizing complex $R^{\cdot}$. Then the following are equivalent:

(1) A is AS-Gorenstein;

(2) $\mathrm{id}_A\, A < \infty$;

(3) $\mathrm{pd}_A\, R^{\cdot} < \infty$;

(4) For any $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Gr}\, A)$, $\mathrm{pd}_A\, X < \infty$ if and only if $\mathrm{id}_A\, X < \infty$.

Proof. (1) $\Rightarrow$ (4). Suppose that $M \in \mathrm{gr}\, A$ with $\mathrm{pd}_A M < \infty$. Then M has a finitely generated free resolution of finite length. Since each term of the resolution has finite injective dimension, so does M. By induction on the amplitude, it is easy to see that $\mathrm{pd}_A\, X < \infty$ implies $\mathrm{id}_A\, X < \infty$ for any $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Gr}\, A)$.

On the other hand, suppose first that $M \in \mathrm{gr}\, A$ with $\mathrm{id}_A M < \infty$. There is a convergent spectral sequence

$$E_2^{p,q} = \underline{\mathrm{Ext}}_A^p(\underline{\mathrm{Ext}}_{A^\circ}^{-q}(k, A), M) \Rightarrow \mathrm{Tor}_{-p-q}^A(k, M).$$

Since $\mathrm{id}_A A < \infty$, $E_2^{p,q} = 0$ for $q \ll 0$. Hence $\mathrm{Tor}_n^A(k, M) = 0$ for $n \gg 0$ and so $\mathrm{pd}_A M < \infty$. Now suppose in general that $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Gr}\, A)$ with $\mathrm{id}_A X < \infty$. We have to show that $\mathrm{pd}_A X < \infty$. Let $F^{\cdot} \cong X$ be a finitely generated free resolution of X and $s = \inf X$. It suffices to prove the s-th syzygy of $F^{\cdot}$ has finite projective dimension. Since $\mathrm{id}_A A < \infty$ and $\mathrm{id}_A X < \infty$, any finitely generated free module has finite injective dimension, and so the s-th syzygy of $F^{\cdot}$ has finite injective dimension. The claim follows from the module case.

(4) $\Rightarrow$ (3). Since $\mathrm{id}_A R^{\cdot} < \infty$, it follows that $\mathrm{pd}_A R^{\cdot} < \infty$.

(3) $\Rightarrow$ (2). By lemma 3.3, $\omega_A = H^{-d}(R^{\cdot})$ is a balanced dualizing module of A, where $\mathrm{pd}_A R^{\cdot} = d$ and $\mathrm{pd}_A(\omega_A) < \infty$. Since $\mathrm{depth}_A(\omega_A)$ =$\mathrm{depth}_A A$, it follows from Auslander-Buchsbaum formula (Theorem 2.7) that $\mathrm{pd}_A(\omega_A) = 0$. So ω_A is free. Since $\mathrm{id}_A(\omega_A) < \infty$, it follows that $\mathrm{id}_A A < \infty$.

(2) $\Rightarrow$ (1). Since A has a balanced dualizing complex, A satisfies the χ-condition. Since $\mathrm{id}_A A < \infty$, $c = \sup\{i \,|\, \underline{\mathrm{Ext}}_A^i(k, A) \neq 0\} < \infty$. It follows from the double Ext spectral sequence

$$E_2^{p,q} = \underline{\mathrm{Ext}}_A^p(\underline{\mathrm{Ext}}_{A^\circ}^{-q}(k, A), A) \Rightarrow \begin{cases} 0 & p+q \neq 0 \\ k & p+q = 0, \end{cases}$$

that $d := \mathrm{depth}_{A^\circ} A < \infty$. Since A° satisfies the χ-condition, it follows from the definition of c that $E_2^{c,-d} = E_\infty^{c,-d} \neq 0$. So $c = d$, that is,

$$\mathrm{depth}_{A^\circ} A = \sup\{i \,|\, \underline{\mathrm{Ext}}_A^i(k, A) \neq 0\}.$$

Since $\mathrm{id}_A A < \infty$, by Lemma 3.4 $\mathrm{pd}_{A^\circ} R^{\cdot} < \infty$. Similar to (3) $\Rightarrow$ (2), we have $\mathrm{id}_{A^\circ} A < \infty$. Thus by the left-right symmetric version of the above spectral sequence,

$$\mathrm{depth}_A A = \sup\{i \,|\, \underline{\mathrm{Ext}}_{A^\circ}^i(k, A) \neq 0\}.$$

Hence

$$\mathrm{depth}_A A = \sup\{i \,|\, \underline{\mathrm{Ext}}_A^i(k, A) \neq 0\} = \mathrm{depth}_{A^\circ} A = \sup\{i \,|\, \underline{\mathrm{Ext}}_{A^\circ}^i(k, A) \neq 0\}.$$

Therefore A is AS Gorenstein. $\square$

The idea in (2) $\Rightarrow$ (1) originates from [SZ, Theorem 3.8]. The above result also tells us that if A is a Noetherian connected graded k-algebra with a balanced dualizing complex, then A is left AS-Gorenstein if and only if A is right AS-Gorenstein. This is [Jo5, Corollary 4.6]. Note that the first condition in Theorem 3.5 is two-sided, while the others are one-sided. In particular, when A has a balanced dualizing complex, then A has finite left injective dimension if and only if that A has finite right injective dimension.

4. Castelnuovo-Mumford regularity and AS-Gorenstein algebras

In this section, we first recall the definitions and some facts of Castelnuovo-Mumford regularity and Ext-regularity. Then we prove any Koszul standard AS-Gorenstein algebra is AS-regular (see Definition 4.8). Note A is a Noetherian connected graded k-algebra as always in this article.

We fix the following conventions: $\inf\{\varnothing\} = +\infty$, $\inf\{\mathbb{Z}\} = -\infty$.

Definition 4.1. [Jo4, 2.1] For any $X^{\cdot} \in \mathrm{D}(\mathrm{Gr}\, A)$, the Castelnuovo-Mumford regularity of $X^{\cdot}$ is defined to be

$$\mathrm{CM.reg}\ X^{\cdot} = \inf\{p \in \mathbb{Z} \mid \mathrm{H}^i_{\mathfrak{m}}(X^{\cdot})_{>p-i} = 0, \forall\ i \in \mathbb{Z}\}.$$

If A has a balanced dualizing complex and $0 \ncong X^{\cdot} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Gr}\ A)$, then $\mathrm{CM.reg}\ X^{\cdot} \neq -\infty$ by the local duality theorem (Theorem 2.11) and $\mathrm{CM.reg}\ X^{\cdot} \neq +\infty$ by Theorem 2.10 (see [Jo4, Observation 2.3]). Moreover, we have $\mathrm{CM.reg}\ ({}_AA) = \mathrm{CM.reg}\ (A_A)$ by [VdB, Corollary 4.8].

Remark 4.2. Römer used the notion of local-regularity in [Rö, 1.1]. For any $M \in \mathrm{Gr}\, A$, $\mathrm{reg}^L_A(M) = \inf\{p \in \mathbb{Z} \mid \mathrm{H}^i_{\mathfrak{m}}(M)_{>p-i} = 0, \forall\ i \geq 0\}$ is called the local-regularity of M. This is identical as the definition of Castelnuovo-Mumford regularity above.

Example 4.3. Since ${}_Ak$ is $\mathfrak{m}$-torsion, we have

$$\mathrm{H}^i_{\mathfrak{m}}({}_Ak) = \begin{cases} 0 & i \neq 0 \\ k & i = 0. \end{cases}$$

By definition 4.1, it is easy to see that $\mathrm{CM.reg}\ {}_Ak = 0$.

Definition 4.4. [Jo4, 2.2] For any $X^{\cdot} \in \mathrm{D}(\mathrm{Gr}\ A)$, the Ext-regularity of $X^{\cdot}$ is defined to be

$$\mathrm{Ext.reg}\ X^{\cdot} = \inf\{p \in \mathbb{Z} \mid \underline{\mathrm{Ext}}^i_A(X^{\cdot}, k)_{<-p-i} = 0, \forall\ i \in \mathbb{Z}\}.$$

Remark 4.5. There is also a notion of regularity defined by Tor-group in literature, which is called the Tor-regularity as in [Rö, 1.1] by Römer. For any $M \in \mathrm{Gr}\, A$, $\mathrm{Tor.reg}_A(M)$ or $\mathrm{reg}^T_A(M) = \inf\{p \in \mathbb{Z} \mid \mathrm{Tor}^A_i(k_A, M)_{>p+i} = 0, \forall\ i \geq 0\}$ is called the Tor-regularity of M. If $M \in \mathrm{gr}\, A$, then as graded k-vector spaces,

$$\underline{\mathrm{Ext}}^j_A(M, k)' \cong \mathrm{Tor}^A_j(k_A, M).$$

Hence $\mathrm{Ext.reg}\ M = \mathrm{reg}^T_A(M)$, i.e., the Tor-regularity is the same as the Ext-regularity for any $M \in \mathrm{gr}\, A$.

For all $0 \ncong X^{\cdot} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Gr}\, A)$, $\mathrm{Ext.reg}\ X^{\cdot} \neq -\infty$. If $\mathrm{Ext.reg}\ X \leq p$ and $F^{\cdot} \cong X^{\cdot}$ is a minimal free resolution of $X^{\cdot}$, then the generators of F^i are concentrated in degree $\leq p - i$. It is possible that $\mathrm{Ext.reg}\ X^{\cdot} = \infty$. Note that $\mathrm{Ext.reg}\ ({}_Ak) = 0$ if and only if A is Koszul, i.e., ${}_Ak$ has a linear free resolution. It was conjectured in [AE] and proved in [AP] that a commutative connected graded k-algebra generated in degree 1 is Koszul if and only if that $\mathrm{Ext.reg}\ k < \infty$.

By using the minimal free resolution, we see that Ext.reg $({}_Ak) =$ Ext.reg (k_A).

Example 4.6. Since ${}_AA$ is free,

$$\underline{\mathrm{Ext}}^i_A(A,k) = \begin{cases} 0 & i \neq 0 \\ k & i = 0. \end{cases}$$

By Definition 4.4, Ext.reg ${}_AA = 0$.

The following result was proved in [Jo4, Theorem 2.5, 2.6], which plays a key role in this article.

Theorem 4.7. Let A be a Noetherian connected graded k-algebra with a balanced dualizing complex. Given any $X^{\cdot} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Gr}\,A)$ with $X^{\cdot} \not\cong 0$. Then

$$-\,\mathrm{CM.reg}\ A \leq \mathrm{Ext.reg}\ X^{\cdot} - \mathrm{CM.reg}\ X^{\cdot} \leq \mathrm{Ext.reg}\ k.$$

If further, A is Koszul and CM.reg $A = 0$, then CM.reg $X^{\cdot} =$ Ext.reg $X^{\cdot}$ for any $0 \not\cong X^{\cdot} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Gr}\ A)$.

A left AS-Gorenstein algebra in the Definition 3.1 is sometimes called of type (d,l). If A is left AS-Gorenstein of type (d,l) and right AS-Gorenstein, say of type (d',l'), then it is well-known that $d = d'$ and $l = l'$.

Definition 4.8. Let A be a Noetherian connected graded k-algebra.

(1) A is called *standard* AS-Gorenstein if A is an AS-Gorenstein algebra with $l = d$.

(2) A is called AS-regular (Artin-Schelter regular) if A is AS-Gorenstein and A has finite global dimension.

It is an easy fact that any Koszul AS-regular algebra is standard (see 4.11). However, a Koszul AS-Gorenstein algebra is not always standard, e.g. $A = k[x]/(x^2)$ is a Noetherian Koszul connected graded k-algebra which has infinite global dimension. Moreover, $\mathrm{id}_A(A) = 0$ and $\underline{\mathrm{Ext}}^0_A(k,A) \cong k(-1)$. Thus, A is a non-standard AS-Gorenstein algebra with $l = -1$ and $d = 0$. In the final part of this section, we prove that any Koszul standard AS-Gorenstein algebra is AS-regular.

Lemma 4.9. Let A be an AS-Gorenstein algebra of type (d,l). Then

$$\mathrm{CM.reg}\ \mathrm{A} = d - l.$$

Proof. Let

$$0 \longrightarrow {}_AA \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

be a minimal injective resolution of ${}_AA$. Then $\underline{\mathrm{Ext}}^i_A(k,A) = \underline{\mathrm{Hom}}_A(k,I^i)$. By Definition 3.1,

$$\underline{\mathrm{Ext}}^i_A(k,A) = \begin{cases} 0 & i \neq d \\ k(l) & i = d, \end{cases}$$

where $d = \mathrm{id}_A(A)$.

Since $\Gamma_{\mathfrak{m}}(I^i) \cong E(\underline{\mathrm{Hom}}_A(k, I^i))$, then

$$\mathrm{H}^i_{\mathfrak{m}}(A) \cong \begin{cases} 0 & i \neq d \\ A'(l) & i = d. \end{cases}$$

By Definition 4.1, it is obvious that CM.reg A $= d - l$. □

Corollary 4.10. Let A be a Koszul standard AS-Gorenstein algebra. Then for any $X^{\cdot} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{Gr}\ A)$ with $X^{\cdot} \ncong 0$,

$$\text{CM.reg X} = \text{Ext.reg X}.$$

Proof. Direct from Theorem 4.7 and Lemma 4.9 □

Theorem 4.11. Let A be a Noetherian connected graded k-algebra. If A is Koszul, then the following statements are equivalent:

(1) A is AS-regular;

(2) A is standard AS-Gorenstein.

Proof. (2) $\Rightarrow$ (1). It suffices to prove that $\mathrm{pd}_A\ k < \infty$. Since A is a Koszul algebra, ${}_Ak$ has a minimal free resolution

$$F.: \quad \cdots \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_0 \longrightarrow k \longrightarrow 0,$$

where $F_i = A(-i)^{\beta_i}$. Suppose that $\mathrm{id}_A A = d$. We claim that the d-th syzygy $Z_d(F.)$ of $F.$ is 0. Assume on the contrary that $Z_d(F.) \neq 0$. Note that

$$\cdots \longrightarrow F_{d+2} \longrightarrow F_{d+1} \longrightarrow Z_d(F.) \longrightarrow 0$$

is a minimal free resolution of $Z_d(F.)$. Then it is easy to see that Ext.reg $Z_d(F.) = d+1$.

Since $\underline{\mathrm{Ext}}^i_A(k, A) = 0$ for $i \neq d$, $\underline{\mathrm{Ext}}^i_A(Z_{d-1}(F.), A) = 0$ for $i \neq 0$. By [Ye, Corollary 4.10] and [Jo1, Theorem 1.2], $A_\alpha(-d)[d]$ is a balanced dualizing complex over A, where α is an automorphism of A as a graded k-algebra. It follows from the local duality theorem (Theorem 2.11) that $\mathrm{H}^i_{\mathfrak{m}}(Z_{d-1}(F.)) = 0$ for $i \neq d$. Since A is AS-Gorenstein, $\mathrm{H}^i_{\mathfrak{m}}(A) = 0$ for $i \neq d$. It follows from $\mathrm{lcd}(A) = d$ and the short exact sequence

$$0 \longrightarrow Z_d(F.) \longrightarrow F_d \longrightarrow Z_{d-1}(F.) \longrightarrow 0$$

that $\mathrm{H}^i_{\mathfrak{m}}(Z_d(F.)) = 0$ for $i \neq d$ and the following sequence is exact:

$$0 \longrightarrow \mathrm{H}^d_{\mathfrak{m}}(Z_d(F.)) \longrightarrow \mathrm{H}^d_{\mathfrak{m}}(F_d) \longrightarrow \mathrm{H}^d_{\mathfrak{m}}(Z_{d-1}(F.)) \longrightarrow 0. \tag{4.1}$$

Since A is a standard AS-Gorenstein algebra, CM.reg $A = 0$ by Lemma 4.9. Hence CM.reg $F_d = d$ by Definition 4.1. Therefore $\mathrm{H}^d_{\mathfrak{m}}(Z_d(F.))_{>0} = \mathrm{H}^d_{\mathfrak{m}}(F_d)_{>0} = 0$ by (4.1). This implies that CM.reg $Z_d(F.) \leq d$, which contradicts to that Ext.reg $Z_d(F.) = d+1$ by Corollary 4.10. Hence $Z_d(F.) = 0$, which means that $\mathrm{pd}_A\ k < \infty$.

(1) $\Rightarrow$ (2). Suppose that A is a Koszul AS-regular k-algebra with global dimension d. Let $L.: 0 \longrightarrow L_d \longrightarrow \cdots \longrightarrow L_0 \longrightarrow k \longrightarrow 0$, where $L_i = A(-i)^{\beta_i}$, be a minimal free resolution of ${}_Ak$. Since

$$\underline{\mathrm{Ext}}^i_A(k, A) = \begin{cases} 0 & i \neq d \\ k(l) & i = d \end{cases}$$

for some $l \in \mathbb{Z}$, $\mathrm{Hom}_A(L., A)$ is a minimal free resolution of $k_A(l)$. It follows from the Koszulity of A that $d = l$. Hence A is standard. □

5. Castelnuovo-Mumford regularity and AS-regular algebras

If A is a polynomial algebra with standard grading, Eisenbud and Goto [EG] proved that $\mathrm{CM.reg}\ M = \mathrm{Tor.reg}\ M$ for all non-zero finitely generated graded A-modules. Recently, Römer proved that the converse is true [Rö, Theorem 4.1]. We copy Römer's result here for the convenience.

Theorem 5.1. Let A be a commutative Noetherian connected graded k-algebra generated in degree 1. The following statements are equivalent:

(i) For all $M \in \mathrm{gr}\ A$, $\mathrm{CM.reg}_A(M) - \mathrm{CM.reg}_A(A) = \mathrm{Ext.reg}_A(M)$;

(ii) For all $M \in \mathrm{gr}\ A$, $\mathrm{Ext.reg}_A(M) = \mathrm{CM.reg}_A(M) + \mathrm{Ext.reg}_A(k)$;

(iii) For all $M \in \mathrm{gr}\ A$, $\mathrm{Ext.reg}_A(M) = \mathrm{CM.reg}_A(M)$;

(iv) A is Koszul and $\mathrm{CM.reg}_A(A) = 0$;

(v) $A = k[x_1, \cdots, x_n]$ is a polynomial ring with standard grading.

In this section, we prove a non-commutative version of [Rö, Theorem 4.1]. We start from the following two lemmas.

Lemma 5.2. Let $X^{\cdot} \in \mathrm{D^b}(\mathrm{Gr}\, A)$ with $h^i(X^{\cdot})_{<-i} = 0$ for any $i \in \mathbb{Z}$. Then for any $i \in \mathbb{Z}$, $\underline{\mathrm{Ext}}^i_A(X^{\cdot}, k)_{>-i} = 0$.

Proof. If $X^{\cdot} \cong 0 \in \mathrm{D^b}(\mathrm{Gr}\ A)$, there is nothing to prove. So we assume $X^{\cdot} \ncong 0$. We prove the assertion by induction on the amplitude $\mathrm{amp}\, X^{\cdot} = \sup X^{\cdot} - \inf X^{\cdot}$.

If $\mathrm{amp}\, X^{\cdot} = 0$, then $h^s(X^{\cdot}) \neq 0$ and $X^{\cdot} \cong h^s(X^{\cdot})[-s] \in \mathrm{D^b}(\mathrm{Gr}\, A)$ for $s = \inf X^{\cdot}$. Since $h^s(X^{\cdot})_{<-s} = 0$, $h^s(X^{\cdot})[-s]$ has a minimal free resolution $F^{\cdot}$, where $(F^i)_{<-i} = 0$ for $i \leq s$ and $F^i = 0$ for $i > s$. Then $\underline{\mathrm{Ext}}^{-i}_A(X^{\cdot}, k)_{>i} = \underline{\mathrm{Hom}}_A(F^i, k)_{>i} = 0$ for $i \leq s$ and $\underline{\mathrm{Ext}}^{-i}_A(X^{\cdot}, k) = 0$ for $i > s$. Thus the assertion holds for $\mathrm{amp}\, X^{\cdot} = 0$.

If $\mathrm{amp}\, X^{\cdot} = n > 0$, we may assume that $X^j = 0$ for either $j < s$ or $j > s + n$, where $s = \inf X^{\cdot}$. Consider the following exact triangle in $\mathrm{D^b}(\mathrm{Gr}\, A)$

$$\mathrm{Ker}(d^s_{X^{\cdot}})[-s] \xrightarrow{\lambda} X^{\cdot} \longrightarrow \mathrm{cone}(\lambda) \longrightarrow \mathrm{Ker}(d^s_{X^{\cdot}})[-s+1].$$

and the induced long exact sequence

$$\cdots \longrightarrow \underline{\mathrm{Ext}}^i_A(\mathrm{cone}(\lambda), k) \longrightarrow \underline{\mathrm{Ext}}^i_A(X^{\cdot}, k) \longrightarrow \underline{\mathrm{Ext}}^i_A(\mathrm{Ker}(d^s_{X^{\cdot}})[-s], k) \longrightarrow \cdots .$$

Since $\operatorname{amp}(\operatorname{Ker}(d^s_{X^\cdot})[-s]) = 0$ and $\operatorname{amp}(\operatorname{cone}(\lambda)) < n$, by induction, it is easy to see that $\underline{\operatorname{Ext}}^i_A(X^\cdot, k)_{>-i} = 0$ for any $i \in \mathbb{Z}$. □

Let A be a ring, and $f : F^\cdot \longrightarrow R^\cdot$ be a morphism of bounded below A-module complexes. Set $s = \inf R^\cdot$. Then f naturally induces a morphism between $F^{\geq s}$ and $R^\cdot$, denoted by $\tilde{f}$. Define a morphism g between $F^{\leq s-1}$ and $\operatorname{cone}(\tilde{f})$ as follows:

$$\begin{array}{ccccccccccc}
F^{\leq s-1} & & \cdots & \longrightarrow & F^{s-2} & \longrightarrow & F^{s-1} & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow g & & & & \downarrow & & \downarrow & & \downarrow & & \\
\operatorname{cone}(\tilde{f}) & & \cdots & \longrightarrow & R^{s-2} & \longrightarrow & F^s \oplus R^{s-1} & \longrightarrow & F^{s+1} \oplus R^s & \longrightarrow & \cdots
\end{array}$$

where

$$g^j = \begin{cases} f^j & j \leq s-2 \\ 0 & j \geq s \\ \begin{pmatrix} d^{s-1}_{F^\cdot} \\ f^{s-1} \end{pmatrix} & j = s-1. \end{cases}$$

Then $g : F^{\leq s-1} \longrightarrow \operatorname{cone}(\tilde{f})$ is indeed a morphism of complexes.

Lemma 5.3. Let the notations be as above. If $f : F^\cdot \longrightarrow R^\cdot$ is a quasi-isomorphism, then g is a quasi-isomorphism.

Proof. It is easy to see that $h^j(F^{\leq s-1}) = 0 = h^j(\operatorname{cone}(\tilde{f}))$ for all $j \neq s-1$, since f is a quasi-isomorphism. We are left to prove that $h^{s-1}(g) : h^{s-1}(F^{\leq s-1}) \longrightarrow h^{s-1}(\operatorname{cone}(\tilde{f}))$ is an isomorphism.

For any $x^{s-1} + \operatorname{Im}(d^{s-2}_{F^\cdot}) \in \operatorname{Ker}(h^{s-1}(g))$, there exists $y^{s-2} \in R^{s-2}$ such that $(d^{s-1}_{F^\cdot}(x^{s-1}), f^{s-1}(x^{s-1})) = (0, d^{s-2}_{R^\cdot}(y^{s-2}))$. Hence $x^{s-1} \in \operatorname{Ker}(d^{s-1}_{F^\cdot}) = \operatorname{Im}(d^{s-2}_{F^\cdot})$. It follows that $h^{s-1}(g)$ is injective.

On the other hand, suppose $(x^s, y^{s-1}) \in \operatorname{Ker}(d^{s-1}_{\operatorname{cone}(\tilde{f})})$, that is $(-d^s_{F^\cdot}(x^s), -f^s(x^s) + d^{s-1}_{R^\cdot}(y^{s-1})) = 0$. Since f is a quasi-isomorphism, there exists $x^{s-1} \in F^{s-1}$ such that $d^{s-1}_{F^\cdot}(x^{s-1}) = x^s$. Then

$$\begin{aligned} (x^s, y^{s-1}) - g^{s-1}(x^{s-1}) &= (x^s, y^{s-1}) - (d^{s-1}_{F^\cdot}(x^{s-1}), f^{s-1}(x^{s-1})) \\ &= (0, y^{s-1} - f^{s-1}(x^{s-1})). \end{aligned}$$

Since $d^{s-1}_{R^\cdot}(y^{s-1} - f^{s-1}(x^{s-1})) = f^s(x^s) - d^{s-1}_{R^\cdot} f^{s-1}(x^{s-1}) = 0$, there exists $x^{s-2} \in R^{s-2}$ such that $d^{s-2}_{R^\cdot}(x^{s-2}) = y^{s-1} - f^{s-1}(x^{s-1})$. Thus

$$h^{s-1}(g)(x^{s-1} + \operatorname{Im}(d^{s-2}_{F^\cdot})) = (x^s, y^{s-1}) + \operatorname{Im}(d^{s-2}_{\operatorname{cone}(\tilde{f})}).$$

It follows that $h^{s-1}(g)$ is surjective. □

The following is the main result in this article, which is a generalization of (iii)⇔(iv)⇔(v) in [Rö, Theorem 4.1] (see also Theorem 5.1) to the non-commutative case.

Theorem 5.4. Let A be a Noetherian connected graded k-algebra with a balanced dualizing complex. Then the following are equivalent:

(1) CM.reg $M =$ Ext.reg M holds for all $M \in \operatorname{gr} A$.

(2) A is Koszul and CM.reg $A = 0$.

(3) A is a Koszul AS-regular k-algebra.

Proof. (3) $\Rightarrow$ (2). By Theorem 4.11, A is standard. It follows from lemma 4.9 that CM.reg $A = 0$.

(2) $\Rightarrow$ (1). Direct from Theorem 4.7.

(1) $\Rightarrow$ (3). Let $R^{\cdot}$ be the balanced dualizing complex over A. Then $R^{\cdot} \cong R\Gamma_{\mathfrak{m}}(A)'$ by [VdB, Theorem 6.3]. By assumption, CM.reg $A =$ Ext.reg $A = 0$, so $\mathrm{H}^i_{\mathfrak{m}}(A)_{>-i} = 0$ for any $i \in \mathbb{Z}$ by Definition 4.1. Thus $h^i(R^{\cdot})_{<-i} = 0$ for any $i \in \mathbb{Z}$. By Lemma 5.2, $\underline{\mathrm{Ext}}^i_A(R^{\cdot}, k)_{>-i} = 0$ for any $i \in \mathbb{Z}$.

On the other hand, since $R^{\cdot}$ is a balanced dualizing complex over A, $R\Gamma_{\mathfrak{m}}(R^{\cdot}) \cong A'$ by Definition 2.9. Therefore, CM.reg $R^{\cdot} = 0$ by Definition 4.1. Again by assumption, Ext.reg $k =$ CM.reg $k = 0$. It follows from Theorem 4.7 that Ext.reg $R^{\cdot} =$ CM.reg $R^{\cdot} = 0$. Thus by Definition 4.4, $\underline{\mathrm{Ext}}^i_A(R^{\cdot}, k)_{<-i} = 0$ for any $i \in \mathbb{Z}$.

Since $R^{\cdot} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(\operatorname{Gr} A)$, $R^{\cdot}$ has a finitely generated minimal free resolution $F^{\cdot} \xrightarrow{\simeq} R^{\cdot}$. Set $s = \inf\{j \mid h^j(R^{\cdot}) \neq 0\}$. Then $s = -\sup\{j \mid \mathrm{H}^j_{\mathfrak{m}}(A) \neq 0\}$. Since both $F^{\geq s}$ and $F^{\leq s-1}$ are minimal free complexes, and $\underline{\mathrm{Ext}}^i_A(R^{\cdot}, k)_{\neq -i} = 0$ for any $i \in \mathbb{Z}$,

$$\text{Ext.reg } F^{\geq s} = \begin{cases} 0 & F^{\geq s} \text{ is not acyclic} \\ -\infty & F^{\geq s} \text{ is acyclic} \end{cases} \tag{5.1}$$

and

$$\text{Ext.reg } F^{\leq s-1} = \begin{cases} 0 & F^{\leq s-1} \text{ is not acyclic} \\ -\infty & F^{\leq s-1} \text{ is acyclic.} \end{cases} \tag{5.2}$$

By the choice of s, $F^s \neq 0$ and thus $\underline{\mathrm{Ext}}^{-s}_A(F^{\geq s}, k) = \underline{\mathrm{Hom}}_A(F^s, k) \neq 0$. Hence $F^{\geq s}$ is not acyclic and Ext.reg $F^{\geq s} = 0$ by (5.1). Let f be the quasi-isomorphism between $F^{\cdot}$ and $R^{\cdot}$. Then f induces naturally a morphism between $F^{\geq s}$ and $R^{\cdot}$, denoted by $\tilde{f}$.

We claim that $\tilde{f}$ is a quasi-isomorphism. If $\tilde{f}$ is not a quasi-isomorphism, then $\operatorname{cone}(\tilde{f}) \ncong 0$ in $\mathrm{D}^{\mathrm{b}}(\operatorname{Gr} A)$. Consider the following exact triangle

$$F^{\geq s} \xrightarrow{\tilde{f}} R^{\cdot} \longrightarrow \operatorname{cone}(\tilde{f}) \longrightarrow F^{\geq s}[1].$$

Since $R\Gamma_{\mathfrak{m}}(R^{\cdot}) \cong A'$ in $\mathrm{D}(\operatorname{Gr} A^e)$, we have the following exact sequences

$$\begin{cases} 0 \to \mathrm{H}^{-1}_{\mathfrak{m}}(\operatorname{cone}(\tilde{f})) \to \mathrm{H}^0_{\mathfrak{m}}(F^{\geq s}) \to \mathrm{H}^0_{\mathfrak{m}}(R^{\cdot}) \to \mathrm{H}^0_{\mathfrak{m}}(\operatorname{cone}(\tilde{f})) \to \mathrm{H}^1_{\mathfrak{m}}(F^{\geq s}) \to 0, \\ \mathrm{H}^{j-1}_{\mathfrak{m}}(\operatorname{cone}(\tilde{f})) \cong \mathrm{H}^j_{\mathfrak{m}}(F^{\geq s}), \quad j \neq 0, 1. \end{cases} \tag{5.3}$$

It follows from the proof of Lemma 5.3 that $\mathrm{cone}(\tilde{f}) \cong h^{s-1}(\mathrm{cone}(\tilde{f}))[1-s]$ in $\mathrm{D^b}(\mathrm{Gr}\, A)$. By the local duality theorem, we have the following isomorphisms

$$\begin{aligned} R\Gamma_{\mathfrak{m}}(\mathrm{cone}(\tilde{f}))' &\cong R\underline{\mathrm{Hom}}_A(\mathrm{cone}(\tilde{f}), R^{\cdot}) \\ &\cong R\underline{\mathrm{Hom}}_A(h^{s-1}(\mathrm{cone}(\tilde{f}))[1-s], R^{\cdot}) \\ &\cong R\underline{\mathrm{Hom}}_A(h^{s-1}(\mathrm{cone}(\tilde{f})), R^{\cdot})[s-1]. \end{aligned}$$

By taking the 0^{th} cohomology modules, we have the following isomorphism,

$$\mathrm{H}^0_{\mathfrak{m}}(\mathrm{cone}(\tilde{f}))' \cong \underline{\mathrm{Ext}}^{s-1}_A(h^{s-1}(\mathrm{cone}(\tilde{f})), R^{\cdot}).$$

Since $h^j(R^{\cdot}) = 0$ for $j < s$, we have $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{cone}(\tilde{f})) = \underline{\mathrm{Ext}}^{s-1}_A(h^{s-1}(\mathrm{cone}(\tilde{f})), R^{\cdot}) = 0$. Since CM.reg $F^{\geq s} =$ Ext.reg $F^{\geq s} = 0$, $\mathrm{H}^{j-1}_{\mathfrak{m}}(\mathrm{cone}(\tilde{f}))_{>-j} = 0$ for any $j \neq 1$ by (5.3). Thus CM.reg $(\mathrm{cone}(\tilde{f})) \leq -1$ by Definition 4.1. However, by Lemma 5.3, $F^{\leq s-1}$ is a minimal free resolution for $\mathrm{cone}(\tilde{f})$. Hence Ext.reg $(\mathrm{cone}(\tilde{f})) = 0$ by (5.2), which contradicts to CM.reg $(\mathrm{cone}(\tilde{f})) \leq -1$. Thus we have proved that $\mathrm{cone}(\tilde{f})$ is acyclic.

Since $F^{\leq s-1}$ is a minimal free resolution for $\mathrm{cone}(\tilde{f})$, then $F^j = 0$ for $j \leq s-1$, which means that the projective dimension of $R^{\cdot}$ is finite. By Theorem 3.5, A is AS-Gorenstein. It follows from Lemma 4.9 and Theorem 4.11 that A is AS-regular. □

Remark 5.5. It is easy to show that (i)⇔(ii) in Theorem 5.1 always holds, even in non-commutative case. By Proposition 5.6, CM.reg $M -$ CM.reg $A =$ Ext.reg M holds for all finitely generated module M with $\mathrm{pd}_A M < \infty$. Since there are lots of non-Koszul AS-regular algebras generated in degree 1 (e.g., see [AS]), (i)⇔(iii) does not hold in general.

Proposition 5.6. Let A be a Noetherian connected graded k-algebra with a balanced dualizing complex. If $M \in \mathrm{gr}\ A$ with $\mathrm{pd}_A M < \infty$, then

$$\text{CM.reg } M - \text{CM.reg } A = \text{Ext.reg } M.$$

Proof. The proof in the commutative case ([Rö, Theorem 4.2]) works well in our non-commutative case. □

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